A Carousel

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Problem statement

Derive the dependence of the angle θ on the angular speed of a carousel ω with a radius l and a rope with the length d, as shown on fig. 1

Solution

Gravity F_g acts on the body in the y direction. Also tension (the rope) acts on the body. Since the body is not moving in the plane rotating with the speed ω around the center, the forces are in equilibrium. But this moving reference frame is not inertial and there is therefore virtual force going out from the center $F_o = \frac{mv^2}{r} = mr\omega^2$. As fig. 1 shows, the angle θ is given

$$\begin{aligned} \frac{F_g}{F_o} &= \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \\ \frac{mg}{mr\omega^2} = \frac{g}{r\omega^2} &= \frac{\cos\theta}{\sin\theta}. \end{aligned}$$

In this case the distance from the center r goes from the triangle

$$r = l + d\sin\theta.$$

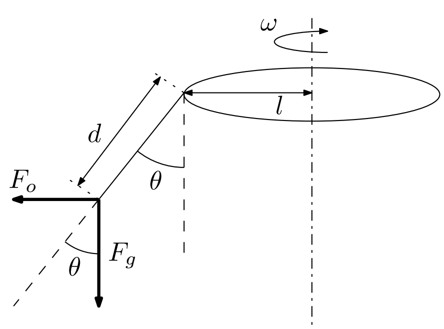

Figure 1: Carousel and the notation of quantities.

Thus the equation becomes

$$\frac{g}{(l + d\sin\theta)\,\omega^2} = \frac{\cos\theta}{\sin\theta}$$

Let's substitute $\sin\theta = t$, $\cos\theta = \sqrt{1-t^2}$, $A = \frac{g}{\omega^2}$

$$\begin{aligned}
\frac{At}{(l+dt)} &= \sqrt{1-t^2} \\
A^2t^2 &= \left(l^2 + d^2t^2 + 2tld\right)\left(1-t^2\right) \\
A^2t^2 &= l^2 + d^2t^2 + 2tld - t^2l^2 - d^2t^4 - 2t^3ld \\
0 &= d^2t^4 + 2t^3ld - t^2\left(d^2 - l^2 - A^2\right) - 2tld - l^2 \\
0 &= t^4 + 2t^3\frac{l}{d} - t^2\left(1 - \frac{l^2}{d^2} - \frac{A^2}{d^2}\right) - 2t\frac{l}{d} - \frac{l}{d}
\end{aligned}$$

This is a quartic equation, which can be solved using Cardano's formula. Let's write $A' = \frac{A}{d}$ and $\delta = \frac{l}{d}$.

$$0 = t^4 + 2\delta t^3 - t^2\left(1 - \delta^2 - A'^2\right) - 2\delta t - \delta$$

For our example it is, however, more pleasant to solve it numerically.[1]

Approximation for small angles

For small angular velocities it is often advantageous to use the approximation $\sin x = x$, $\cos x = 1$. Then the equation becomes

$$\begin{aligned}
\frac{(l + d\theta)\,\omega^2}{g} &= \theta \\
\theta &= \frac{\omega^2 l}{g - d\omega^2}
\end{aligned}$$

[1] Algebraic solution to quartic can be found on, e. g., `http://mathworld.wolfram.com/QuarticEquation.html`.